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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

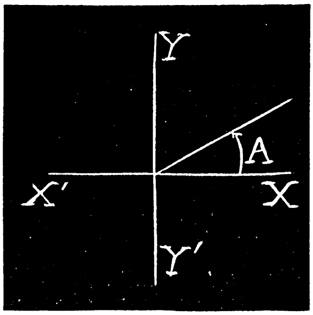
VOL. XVIII. OCTOBER, 1911. NO. 10.

# REDUCTION OF THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE TO THE FUNCTIONS OF THE ANGLES IN A SMALL INTERVAL.

By DR. G. A. MILLER.

In view of the great practical importance of the reduction of the trigonometric functions of an angle to the similar functions of an angle in an interval which does not exceed 90°, it may be of interest to present a method of reduction which is simple and has some points of novelty even if this method is not decidedly superior to the one commonly employed in the text-books on trigonometry. The method has, however, the strong point in its favor that it is applicable to many other problems and exhibits deep contact which does not appear so clearly in the methods commonly employed.

For simplicity of statement we shall always assume that the angle $A$ under consideration has its vertex at the origin of a rectangular system of coördinates and that the initial line coincides with the positive part of the $x$-axis. It is easy to see that we may find the geometric* angle which corresponds to the complement of $A$ by reflecting the terminal line of $A$ on the bisector of the first and third quadrants.



Similarly, we may find the supplement of $A$ by reflecting the terminal line of $A$ on the $y$-axis.

If we perform the operation of taking the complement ($c$) and of taking the supplement ($s$) of the finite angle $A$ and of the resulting angles, we obtain, in general, eight different geometric angles. We never obtain more than eight such angles, and we always obtain exactly eight except when the terminal line of $A$ lies either on one of the coördinate axes or on a bisector of one of the

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[*] It is well known that each geometric angle has an infinite number of different measures. These measures will be called the analytic angles corresponding to the given geometric angle. Hence $\alpha$ and $\alpha+360°$ represent different analytic angles but they represent the same geometric angles. The trigonometric functions are based upon the geometric angles.

quadrants. For instance, when $A=20°$ the following eight angles form a complete set of conjugates with respect to the operations $c$ and $s$:

$$20°, 70°, 110°, 160°, 200°, 250°, 290°, 340°.$$

That is, we obtain no new geometric angle by performing the operations $c$ and $s$ upon any one of these eight angles, and if we perform these operations successively upon any one of them and upon the angles which result we obtain *all* of them.

In general, if we perform upon the angle $A$, and upon the resulting angles, the operations $c$ and $s$ we obtain the following eight angles:

$$A,\ 90°-A,\ 90°+A,\ 180°+A,\ 270°-A,\ 270°+A,\ 180°-A.$$

It may be observed that these eight angles are similarly situated as regards the quadrants, and that each one of the octants into which the plane is divided by the coördinate axes and the bisectors of the quadrants contains one and only one of these angles if $A$ is not on the boundary of an octant. If $A$ is on the boundary of such an octant the given eight angles reduce to four distinct geometric angles which include either the beginning line or the terminal line of every octant. This proves the following fundamental theorem:

*By performing successively upon any angle and upon the resulting angles the operations of getting its complement and its supplement we always obtain a set of angles which include one and only one geometric angle in the interval from 0° to 45°, and all of the angles of this set can be obtained from each one of them by means of these operations.*

From the preceding theorem it results directly that if we know all the trigonometric functions of the angles in the interval from 0° to 45°, and if we also know how to find these functions of the complement and of the supplement of $A$ from the functions of $A$, we can find the trigonometric functions of any angle whatsoever. The finding of the trigonometric functions of any angle is therefore reduced to three tables, as follows:

I. Table giving the trigonometric functions of angles in the interval from 0° to 45°.
II. $\sin A=\cos(90°-A)$, $\cot A=\tan(90°-A)$,
$\cos A-\sin(90°-A)$, $\sec A=\csc(90°-A)$,
$\tan A=\cot(90°-A)$, $\csc A=\sec(90°-A$.
III. $\sin A=\sin(180°-A)$, $\cot A=-\cot(180°-A)$,
$\cos A=-\cos(180°-A)$, $\sec A=-\sec(180°-A)$,
$\tan A=-\tan(180°-A)$, $\csc A=\csc(180°-A)$.

While these three tables are sufficient to find the trigonometric functions of any angle, the other tables generally given in text-books on trigonometry are desirable as they save time. This presentation of the matter is offered mainly on account of its theoretic interest, and because it connects this elementary process in trigonometry with the extensive subject of group theory. In fact, the operations denoted by $c$ and $s$ generate a group of order 8 known as the octic group or the group of the square. From this fact it results directly that the operations $c$ and $s$ could be replaced by any pair of generators of this octic group. In particular, we could replace Table III by one giving the functions of $A$ in terms of functions of $-A$, without changing tables I and II.

The point of view developed above has other advantages as it directs attention to the fact that we replace Table I by one covering a very much smaller interval provided we replaced the other two tables by suitable tables. In fact, Table I could be diminished indefinitely, but such a reduction would have no practical bearing. To indicate changes along this line, which have only theoretical interest, we may observe that Table I could be replaced by one covering only the interval from 0° to 22½°, by assuming Table II and replacing Table III by the following:

$$\sin(45^\circ-x)=\tfrac{1}{2}\sqrt{2}(\cos x-\sin x),\quad \cot(45^\circ-x)=\frac{\cos x+\sin x}{\cos x-\sin x},$$

$$\cos(45^\circ-x)=\tfrac{1}{2}\sqrt{2}(\cos x+\sin x),\quad \sec(45^\circ-x)=\frac{\sqrt{2}}{\cos x+\sin x},$$

$$\tan(45^\circ-x)=\frac{\cos x-\sin x}{\cos x+\sin x},\quad \csc(45^\circ-x)=\frac{\sqrt{2}}{\cos x-\sin x}.$$

In fact, the operations of getting the complement and of subtracting from 45° generate a group of order 16, as may readily be verified,* and by these operations an angle is, in general, transformed into one and into only one angle in each of the sixteen equal divisions of the circumangle, starting from 0°. Hence we can find the trigonometric functions of any angle if we know the functions of all the angles in one of those intervals of 22½° and the operations expressed in the table ending the preceding paragraph, together with those of Table II. There is evidently no limit to similar developments along these lines.

One of the many general methods which includes the special cases given above and illustrates how the interval of the angles whose trigonometric functions are supposed to be known may be reduced indefinitely, is as follows: The group of movements of the regular polygon of $n$ sides is the dihedral group of order $2n$. This group transforms *transitively* each of the

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* *Annals of Mathematics*, Vol. 8 (1907), p. 97.

$2n$ half sides of the regular polygon; that is, it may be represented as a regular substitution group on letters so selected that each letter corresponds to one of these half sides. Hence *the group of movements of the regular polygon contains one and only one operator which transforms a particular half side of the polygon into a given half side.* As this group of movements is dihedral it may be generated by two of its operators of order two.

It is known that subtraction may be regarded geometrically as a reflection of the point representing the subtrahend on the point midway between the minuend and the origin.* If we inscribe in a circle a regular polygon of $2m$ sides, $m>1$, the group of movements of this polygon is clearly generated by a rotation through $\pi$ on the line of symmetry bisecting the first side and a rotation through $\pi$ on the $y$-axis. These rotations are equivalent, respectively, to subtraction from $\pi/m$ and subtraction from $\pi$. If we make $m=2$ we have the group, considered above, formed by the operations of getting the complement and the supplement.

In the second example considered above, $m=4$, and, instead of subtracting from $\pi$ we subtracted from $\frac{1}{2}\pi$, which is evidently always permissible when $m$ is even and greater than 2. As the table of the trigonometric functions may be confined to the interval $\pi/2m$ provided we can deduce the functions of $\pi-A$, and of $\pi/m—A$ from those of $A$, this indicates how the necessary Table I may be reduced theoretically to an interval which is less than any given interval. Practically, this method should probably not be employed beyond $m=2$.

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* *Annals of Mathematics*, Vol. 6 (1905), p. 41.

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# A SET OF INDEPENDENT ASSUMPTIONS FOR PROJECTIVE GEOMETRY.*

By N. J. LENNES, Columbia University.

## INTRODUCTION.

The purpose of the present paper is sufficiently indicated in the title. Incidentally an elementary statement is made of well-understood principles in the foundations of mathematics, and the actual working of these principles is exhibited in treating the set of assumptions for projective geometry. Other sets of assumptions for projective geometry have been given by Veblen and Young† and by the present writer.‡ It is believed that the set of assumptions given in the present paper will lead to a development of Pro-

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* The substance of this paper was presented to the American Mathematical Society in February, 1911.

† Veblen and Young, *Projective Geometry*, Vol. I.

‡ Lennes, "Duality in Projective Geometry," *Annals of Mathematics*, Vol. 11 (1911).